Scalars convected by a 2D incompressible flow

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1 Abstract

We provide a test for numerical simulations, for several two dimensional incompressible flows, that appear to develop sharp fronts. We show that in order to have a front the velocity has to have uncontrolled velocity growth.

2 Introduction

The aim of this paper is to study the possible formation of sharp fronts in finite time for a scalar convected by a two dimensional divergence-free velocity field, with $x = (x_1, x_2) \in R^2$ or R^2/Z^2, and $t \in [0, T)$ with $T \leq \infty$. The

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scalar function $\theta(x,t)$ and the velocity field $u(x,t) = (u_1(x,t), u_2(x,t)) \in R^2$ satisfy the following set of equations

$$\begin{aligned} (\partial_t + u \cdot \nabla)\, \theta &= 0 \qquad (1) \\ \nabla^{\perp} \psi &= u, \end{aligned}$$

where $\nabla_x^{\perp} f = (-\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1})$ for scalar functions f. The function ψ is the stream function.

There are many physical examples where the solutions satisfy the equations above, with an extra equation or operator that relates θ with the velocity field. Examples include; Passive scalars, Unsteady Prandtl equations, 2D incompressible Euler equations, Boussinesq, 2D Ideal Magnetohydrodynamics and the Quasi-geostrophic equation.

In the literature on numerical simulations for the 2D Ideal Magnetohydrodynamics (MHD) a standard candidate for a current sheet formation (see Fig. 1) is when the level sets of the magnetic stream function (represented in (1) by θ) contain a hyperbolic saddle (an X-point configuration). The front is formed when the hyperbolic saddle closes, and becomes two Y-points configuration joined by a current sheet. (See Parker [12], Priest-Titov-Rickard [13], Friedel-Grauer-Marliani [10] and Cordoba-Marliani [8].)

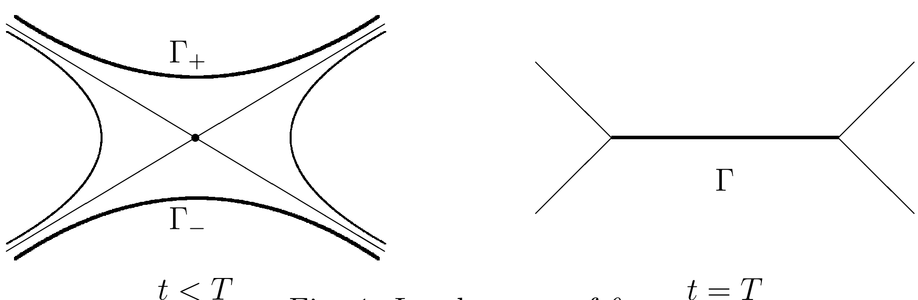

Fig. 1. Level curves of θ.

The same configuration was observed in numerical simulations for the Quasi-geostrophic equation (QG). In this case the geometry of the level sets of the temperature has a hyperbolic structure (See Constantin-Majda-Tabak [4], Okhitani-Yamada [11], Cordoba [6] and Constantin-Nie-Schorghofer [5]). The QG literature discusses X-points, but not Y-points. In the case of Boussinesq there is no mention, on any numerical simulation study, that a possible singularity is due to the closing of a hyperbolic saddle. In the work of Pumir-Siggia [14] there has been observed evidence for a formation of a front in finite time, across which θ varies dramatically, on a cap of a symmetric rising bubble. E-Shu [9] performed numerical simulations with the same initial data

as in [14], which suggest that the thickness of the bubble decreases only exponentially.

The equations for MHD, QG and Boussinesq are as follows

MHD:

$$\begin{aligned}(\partial_t + u \cdot \nabla)\theta &= 0 \\ (\partial_t + u \cdot \nabla)\omega &= \nabla^{\perp}\theta \cdot \nabla(\Delta\theta) \\ u &= \nabla^{\perp}\psi\end{aligned}$$

and initial conditions $\theta(x,0) = \theta_0$ and $u(x,0) = u_0$. The $\nabla^{\perp}\theta$ represents the magnetic field, $\Delta\theta$ represents the current density and $\omega = -\Delta\psi$ the vorticity.

QG:

$$(\partial_t + u \cdot \nabla)\theta = 0$$

$$u = \nabla^{\perp}\psi \quad where \quad \theta = -(-\triangle)^{\frac{1}{2}}\psi$$

and initial condition $\theta(x,0) = \theta_0$. The temperature is represented by θ.

Boussinesq:

$$\begin{aligned}(\partial_t + u \cdot \nabla_x)\,\theta &= 0 \\ (\partial_t + u \cdot \nabla_x)\,\omega &= -\theta_{x_1} \\ u &= \nabla^{\perp}\psi\end{aligned}$$

Again, θ and u are specified at time t=0.

3 Criterion

A singularity can be formed by collision of two particle trajectories. A trajectory X(q,t) is obtain by solving the following ordinary differential equation

$$\begin{aligned}\frac{dX(q,t)}{dt} &= u(X(q,t),t) \\ X(q,0) &= q\end{aligned}$$

Therefore,

$$(X(q,t) - X(p,t))_t \leq |X(q,t) - X(p,t)||\nabla u|_{L^\infty}$$

$$|X(q,t) - X(p,t)| \geq |X(q,0) - X(p,0)| e^{-\int_0^t |\nabla u|_{L^\infty} ds}$$

By this trivial argument; in order to have a collision the quantity $\int_0^t |\nabla u|_{L^\infty} ds$ has to diverge.

A classic criterion for formation of singularities in fluid flows is the theorem of Beale-Kato-Majda (BKM); (see [1]), which improves the estimate described above, and deals with arbitrary singularities, not just collisions. Analogues of the BKM theorem for the above 2-dimensional equations include the following results

For **MHD**,a singularity cannot develop at a finite time T, unless we have

$$\int_0^T sup_x |\omega(x,t)| + sup_x |\triangle_x \theta(x,t)| dt = \infty,$$

where ω denotes the vorticity.(See Caflisch-Klapper-Steele [2].)

For **QG**, a singularity cannot develop at a finite time T, unless we have

$$\int_0^T sup_x |\nabla_x \theta(x,t)| dt = \infty,$$

(See Constantin-Majda-Tabak [4]).

For **Boussinesq**, if a singularity develops at a finite time T then

$$\int_0^T sup_x |\omega(x,t)| dt = \infty \quad and \quad \int_0^T \int_0^t sup_x |\nabla_x \theta(x,s)| ds dt = \infty.$$

(See E-Shu [9].)

See also Constantin-Majda-Tabak [4] and Constantin-Fefferman-Majda [3] for other conditions involving direction fields, that rule out formation of singularities in fluids.

In the case of 2D Euler, a singularity cannot develop at a finite time. From the BKM viewpoint this follows from the fact that ω is advected by the fluid, and therefore $sup_x |\omega(x,t)|$ is independent of t. (See BKM [1].)

Instead of looking at particle trajectories we look at level curves. Because the scalar function θ is convected by the flow, that implies that the level curves are transported by the flow. A possible singular scenario is due to level curves approaching each other very fast which will lead to a fast growth of the gradient of the scalar function. In this paper we present a variant of the BKM criterion for sharp front formation. We provide a test for numerical simulations that appear to develop sharp fronts. The BKM Theorem shows

that the vorticity grows large if any singularity forms; our Theorem 1 shows that the velocity grows large if a sharp front forms.

The theorem we present in this paper was announced in [7].

4 Sharp Fronts

The scalar function θ is convected by the flow, therefore the level curves move with the flow. A sharp front forms when two of these level curves collapse on a single curve. We define two level curves to be two distinct time-dependent arcs $\Gamma_+(t)$, $\Gamma_-(t)$ that move with the fluid and collapse at finite time into a single arc Γ. More precisely, suppose the arcs are given by

$$\Gamma_\pm = \{(x_1, x_2) \in R^2 : x_2 = f_\pm(x, t), x_1 \in [a, b]\} \;\; for \;\; 0 \le t < T, \tag{2}$$

with

$$f_\pm \in C^1([a, b] \times [0, T)) \tag{3}$$

and

$$f_-(x_1, t) < f_+(x_1, t) \;\; for \;\; all \;\; x_1 \in [a, b], \;\; t \in [0, T). \tag{4}$$

We call the length b-a of the interval [a,b] the length of the front. The assumption that $\Gamma_\pm(t)$ move with the fluid means that

$$u_2(x_1, x_2, t) = \frac{\partial f_\pm}{\partial x_1}(x_1, t) \cdot u_1(x_1, x_2, t) + \frac{\partial f_\pm}{\partial t}(x_1, t) \;\; at \;\; x_2 = f_\pm(x_1, t). \tag{5}$$

This holds in particular for level curves of scalar functions g(x,t) that satisfy $(\partial_t + u \cdot \nabla_x)\, g = 0$. The collapse of $\Gamma_\pm(t)$ into a single curve Γ at time T means here simply that

$$lim_{t \to T^-}(f_+(x_1, t) - f_-(x_1, t)) = 0 \;\; for \;\; all \;\; x_1 \in [a, b]. \tag{6}$$

and $f_+(x_1, t) - f_-(x_1, t)$ is bounded for all $x_1 \in [a, b]$, $t \in [0, T)$.

When (2), (3), (4), (5) and (6) hold, then we say that the fluid forms a **sharp front** at time T.

The standard candidates for a singularity for MHD and QG are described by the definition given for a sharp front. We investigate the possible formation of a sharp front.

The following assumption will allow us to rule out formation of sharp fronts. We say that the fluid has **controlled velocity growth** if we have

$$\int_0^T sup\{|u(x_1,x_2,t)| : x_1 \in [a,b], f_-(x_1,t) \le x_2 \le f_+(x_1,t)\}dt < \infty. \qquad (7)$$

If (7) fails, then we say that the fluid has **uncontrolled velocity growth**.

Lemma 1. *Let θ be a smooth solution of Eq.1 defined for $t \in [0,T)$. Assume there is a* ***sharp front*** *at time T. Then*

$$\begin{aligned}\left(\frac{d}{dt}\right)\left(\int_a^b [f_+(x_1,t) - f_-(x_1,t)]dx_1\right) &= \psi(a, f_+(a,t),t) - \psi(a, f_-(a,t),t) \\ &+ \psi(b, f_-(b,t),t) - \psi(b, f_+(b,t),t). \qquad (8)\end{aligned}$$

Proof: Take the derivative of the stream function with respect to x_1 along an arc $\Gamma_\pm(t)$

$$\frac{\partial\psi(x_1, f_\pm(x_1,t),t)}{\partial x_1} = u_2(x_1, f_\pm(x_1,t),t) - \frac{\partial f_\pm}{\partial x_1} u_1(x_1, f_\pm(x_1,t),t) \qquad (9)$$

by combining (9) and (5) we obtain

$$\frac{\partial\psi(x_1, f_\pm(x_1,t),t)}{\partial x_1} = \frac{\partial f_\pm}{\partial t}(x_1,t) \qquad (10)$$

Expression (8) follows from integrating (10) with respect to x_1 between a and b.

Theorem 1. *Let u(x,t) be a divergence-free velocity field, with controlled velocity growth. Then a* ***sharp front*** *cannot develop at time T.*

Proof: Assume there is a sharp front at time T. We define

$$A(t) = \int_{\tilde{a}(t)}^{\tilde{b}(t)} [f_+(x_1,t) - f_-(x_1,t)]dx_1$$

where

$$\tilde{a}(t) = a + \int_t^T sup\{|u(x_1,x_2,s)| : x_1 \in [a,b], f_-(x_1,s) \le x_2 \le f_+(x_1,s)\}ds$$

and

$$\tilde{b}(t) = b - \int_t^T sup\{|u(x_1, x_2, s)| : x_1 \in [a, b], f_-(x_1, s) \leq x_2 \leq f_+(x_1, s)\} ds$$

There is controlled velocity growth, therefore there exists $t^* \in [0, T)$ such that $\tilde{a}(t) \in [a, b]$ and $\tilde{b}(t) \in [a, b]$ for all $t \in [t^*, T)$.

We take the derivative of A(t) with respect to time

$$\frac{dA(t)}{dt} = sup|u| \cdot \delta(\tilde{b}, t) + sup|u| \cdot \delta(\tilde{a}, t) + \int_{\tilde{a}(t)}^{\tilde{b}(t)} \frac{\partial}{\partial t}[f_+(x_1, t) - f_-(x_1, t)] dx_1.$$

where $sup|u| = sup\{|u(x_1, x_2, t)| : x_1 \in [a, b], f_-(x_1, t) \leq x_2 \leq f_+(x_1, t)\}$ and $\delta(z, t) = f_+(z, t) - f_-(z, t)$.

Using the definition of the stream function, the mean value theorem and (8), it is easy to check that $\frac{dA(t)}{dt} > 0$ for $t > t^*$. This contradicts (6) by the dominated convergence theorem.

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